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On Theta-Functions with Complex Characteristics.

BY THOMAS CRAIG, *Johns Hopkins University.*

Prym in his memoir "Untersuchungen über die Riemann'sche Thetaformel und die Riemann'sche Charakteristikentheorie" considers characteristics

$$\begin{pmatrix} g_1 g_2 \dots g_p \\ h_1 h_2 \dots h_p \end{pmatrix}$$

where the quantities g, h are any constants instead of being as usual integers. If the quantities g and h are rational fractions with a common denominator, say r, then it is shown that there are in all r^{2p} different functions. Krazer in his Habilitationsschrift considers the problem of finding the relations connecting these r^{2p} different theta-functions, limiting himself however to the case of $p = 1$, $r = 3$. In what follows I have just begun a study of the theta-functions corresponding to the case of a characteristic made up of complex quantities. I have called the functions Ξ-functions to distinguish them from the ordinary theta-functions when the characteristic is made up of integers.

The theta-function of p variables is defined by the equation

$$\vartheta\begin{pmatrix} l_1, & l_2 & \dots & l_p \\ \lambda_1, & \lambda_2 & \dots & \lambda_p \end{pmatrix}(v_1, v_2 \dots v_p) = \Sigma \exp. \begin{pmatrix} n_1 + l_1, & n_2 + l_2 & \dots & n_p + l_p \\ v_1 + \lambda_1, & v_2 + \lambda_2 & \dots & v_p + \lambda_p \end{pmatrix},$$

where

$$\begin{pmatrix} n_1 + l_1, & n_2 + l_2 & \dots & n_p + l_p \\ v_1 + \lambda_1, & v_2 + \lambda_2 & \dots & v_p + \lambda_p \end{pmatrix} = \tfrac{1}{4}(*)(n_1 + l_1, n_2 + l_2 \dots n_p + l_p)^2$$
$$+ \tfrac{1}{2}\pi i\{(n_1 + l_1)(v_1 + \lambda_1) + \dots + (n_p + l_p)(v_p + \lambda_p)\}$$

and $(*)$ denotes the constants

$$\begin{matrix} a_{11} a_{12} a_{13} \dots a_{1p} \\ a_{22} a_{23} \dots a_{2p} \\ a_{33} \dots a_{3p} \\ \dots\dots\dots \\ a_{pp}. \end{matrix}$$

It will be understood of course that $a_{rs} = a_{sr}$.

In the case of the theta-functions the characteristic

$$\begin{pmatrix} l_1, & l_2 & \dots & l_p \\ \lambda_1, & \lambda_2 & \dots & \lambda_p \end{pmatrix}$$

may be made up of positive or negative integers, but, as a matter of fact, l and λ are always made equal to either zero or unity. In what follows I propose to examine the functions for which l, λ are complex quantities, say

$$l_r = a_r + ib_r$$
$$\lambda_r = \alpha_r + i\beta_r,$$

and particularly the cases when a, b, α, β take the values zero and unity, and also when $a_r + ib_r$ and $\alpha_r + i\beta_r$ are conjugate complex quantities. I will denote the functions under consideration by the symbol Ξ, that is

$$1. \quad \Xi\begin{pmatrix} l_1, & l_2 \dots l_p \\ \lambda_1, & \lambda_2 \dots \lambda_p \end{pmatrix}(v_1,\ v_2 \dots v_p) = \Sigma \text{ exp.} \begin{pmatrix} n_1 + l_1, & n_2 + l_2 \dots n_p + l_p \\ v_1 + \lambda_1, & v_2 + \lambda_2 \dots v_p + \lambda_p \end{pmatrix}.$$

It will be understood in all that follows, when the symbol Ξ is used, that the letters l and λ denote complex quantities, and when ϑ is used that they denote positive or negative integers, or, as usual, simply zero and unity. The convergency of the Ξ-functions is secured as in the case of the ϑ-functions: "the parameters a_{rs} may be real or imaginary, but they must be such that reducing each of them to its real part the resulting function $(*\between n_1, n_2 \dots n_p)^2$ is invariable in its sign, and negative for all real values of n_r".* I will assume at once that a_r, b_r, α_r, $\beta_r = 0$ or 1; the total number of Ξ-functions is now easily found. Denote by (a, α) the entire set of a's and α's, and by (b, β) the entire set of b's and β's. Assume first that $(b, \beta) = 0$ and give (a, α) all possible values made up of 0 and 1; we have then 2^{2p}, or 4^p functions. Now give (b, β) all possible values made up of zero and unity, while $(a, \alpha) = 0$, and we have again 4^p functions, so that finally we have as the total number of Ξ-functions $(4^p)^2$ or 4^{2p}. Thus the total number of p-tuple Ξ-functions is equal to the square of the total number of ϑ-functions. For example:

Total number of ϑ-functions.		Total number of Ξ-functions.
$p = 1$	4	16
$p = 2$	16	256
$p = 3$	64	4096
$p = 4$	256	65536
etc.	etc.	etc.
. . .	. . .	. . .
$p = p$	2^{2p}	2^{4p}.

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* A Memoir on the Single and Double Theta-Functions, by A. Cayley (page 899), *Phil. Trans.*, 1880.

For the case of $p = 1$ the Ξ-characteristics are as follows:

2.
$$\begin{array}{cccc}
\begin{pmatrix}0\\0\end{pmatrix} & \begin{pmatrix}i\\0\end{pmatrix} & \begin{pmatrix}0\\i\end{pmatrix} & \begin{pmatrix}i\\i\end{pmatrix}\\
\begin{pmatrix}1\\0\end{pmatrix} & \begin{pmatrix}1+i\\0\end{pmatrix} & \begin{pmatrix}1\\i\end{pmatrix} & \begin{pmatrix}1+i\\i\end{pmatrix}\\
\begin{pmatrix}0\\1\end{pmatrix} & \begin{pmatrix}i\\1\end{pmatrix} & \begin{pmatrix}0\\1+i\end{pmatrix} & \begin{pmatrix}i\\1+i\end{pmatrix}\\
\begin{pmatrix}1\\1\end{pmatrix} & \begin{pmatrix}1+i\\1\end{pmatrix} & \begin{pmatrix}1\\1+i\end{pmatrix} & \begin{pmatrix}1+i\\1+i\end{pmatrix}
\end{array}$$

In order to form the characteristics for the case of $p = 2$ we have only to combine the following two tables, the first of which corresponds to $(b, \beta) = 0$, and the second to $(a, \alpha) = 0$,

3.

0 0 0 0	1 0 0 0	0 1 0 0	1 1 0 0
0 0 1 0	1 0 1 0	0 1 1 0	1 1 1 0
0 0 0 1	1 0 0 1	0 1 0 1	1 1 0 1
0 0 1 1	1 0 1 1	0 1 1 1	1 1 1 1

0 0 0 0	i 0 0 0	0 i 0 0	i i 0 0
0 0 i 0	i 0 i 0	0 i i 0	i i i 0
0 0 0 i	i 0 0 i	0 i 0 i	i i 0 i
0 0 i i	i 0 i i	0 i i i	i i i i

We have then for the Ξ characteristics in the case of $p = 2$ the following table:

4.

$\begin{matrix}0 & 0\\ 0 & 0\end{matrix}$	$\begin{matrix}i & 0\\ 0 & 0\end{matrix}$	$\begin{matrix}0 & i\\ 0 & 0\end{matrix}$	$\begin{matrix}i & i\\ 0 & 0\end{matrix}$	$\begin{matrix}0 & 0\\ i & 0\end{matrix}$	$\begin{matrix}i & 0\\ i & 0\end{matrix}$	$\begin{matrix}0 & i\\ i & 0\end{matrix}$	$\begin{matrix}i & i\\ i & 0\end{matrix}$	$\begin{matrix}0 & 0\\ 0 & i\end{matrix}$	$\begin{matrix}i & 0\\ 0 & i\end{matrix}$	$\begin{matrix}0 & i\\ 0 & i\end{matrix}$	$\begin{matrix}i & i\\ 0 & i\end{matrix}$	$\begin{matrix}0 & 0\\ i & i\end{matrix}$	$\begin{matrix}i & 0\\ i & i\end{matrix}$	$\begin{matrix}0 & i\\ i & i\end{matrix}$	$\begin{matrix}i & i\\ i & i\end{matrix}$
$\begin{matrix}1 & 0\\ 0 & 0\end{matrix}$	$\begin{matrix}1+i & 0\\ 0 & 0\end{matrix}$	$\begin{matrix}1 & i\\ 0 & 0\end{matrix}$	$\begin{matrix}1+i & i\\ 0 & 0\end{matrix}$	$\begin{matrix}1 & 0\\ i & 0\end{matrix}$	$\begin{matrix}1+i & 0\\ i & 0\end{matrix}$	$\begin{matrix}1 & i\\ i & 0\end{matrix}$	$\begin{matrix}1+i & i\\ i & 0\end{matrix}$	$\begin{matrix}1 & 0\\ 0 & i\end{matrix}$	$\begin{matrix}1+i & 0\\ 0 & i\end{matrix}$	$\begin{matrix}1 & i\\ 0 & i\end{matrix}$	$\begin{matrix}1+i & i\\ 0 & i\end{matrix}$	$\begin{matrix}1 & 0\\ i & i\end{matrix}$	$\begin{matrix}1+i & 0\\ i & i\end{matrix}$	$\begin{matrix}1 & i\\ i & i\end{matrix}$	$\begin{matrix}1+i & i\\ i & i\end{matrix}$
$\begin{matrix}0 & 1\\ 0 & 0\end{matrix}$	$\begin{matrix}i & 1\\ 0 & 0\end{matrix}$	$\begin{matrix}0 & 1+i\\ 0 & 0\end{matrix}$	$\begin{matrix}i & 1+i\\ 0 & 0\end{matrix}$	$\begin{matrix}0 & 1\\ i & 0\end{matrix}$	$\begin{matrix}i & 1\\ i & 0\end{matrix}$	$\begin{matrix}0 & 1+i\\ i & 0\end{matrix}$	$\begin{matrix}i & 1+i\\ i & 0\end{matrix}$	$\begin{matrix}0 & 1\\ 0 & i\end{matrix}$	$\begin{matrix}i & 1\\ 0 & i\end{matrix}$	$\begin{matrix}0 & 1+i\\ 0 & i\end{matrix}$	$\begin{matrix}i & 1+i\\ 0 & i\end{matrix}$	$\begin{matrix}0 & 1\\ i & i\end{matrix}$	$\begin{matrix}i & 1\\ i & i\end{matrix}$	$\begin{matrix}0 & 1+i\\ i & i\end{matrix}$	$\begin{matrix}i & 1+i\\ i & i\end{matrix}$
$\begin{matrix}1 & 1\\ 0 & 0\end{matrix}$	$\begin{matrix}1+i & 1\\ 0 & 0\end{matrix}$	$\begin{matrix}1 & 1+i\\ 0 & 0\end{matrix}$	$\begin{matrix}1+i & 1+i\\ 0 & 0\end{matrix}$	$\begin{matrix}1 & 1\\ i & 0\end{matrix}$	$\begin{matrix}1+i & 1\\ i & 0\end{matrix}$	$\begin{matrix}1 & 1+i\\ i & 0\end{matrix}$	$\begin{matrix}1+i & 1+i\\ i & 0\end{matrix}$	$\begin{matrix}1 & 1\\ 0 & i\end{matrix}$	$\begin{matrix}1+i & 1\\ 0 & i\end{matrix}$	$\begin{matrix}1 & 1+i\\ 0 & i\end{matrix}$	$\begin{matrix}1+i & 1+i\\ 0 & i\end{matrix}$	$\begin{matrix}1 & 1\\ i & i\end{matrix}$	$\begin{matrix}1+i & 1\\ i & i\end{matrix}$	$\begin{matrix}1 & 1+i\\ i & i\end{matrix}$	$\begin{matrix}1+i & 1+i\\ i & i\end{matrix}$
$\begin{matrix}0 & 0\\ 1 & 0\end{matrix}$	$\begin{matrix}i & 0\\ 1 & 0\end{matrix}$	$\begin{matrix}0 & i\\ 1 & 0\end{matrix}$	$\begin{matrix}i & i\\ 1 & 0\end{matrix}$	$\begin{matrix}0 & 0\\ 1+i & 0\end{matrix}$	$\begin{matrix}i & 0\\ 1+i & 0\end{matrix}$	$\begin{matrix}0 & i\\ 1+i & 0\end{matrix}$	$\begin{matrix}i & i\\ 1+i & 0\end{matrix}$	$\begin{matrix}0 & 0\\ 1 & i\end{matrix}$	$\begin{matrix}i & 0\\ 1 & i\end{matrix}$	$\begin{matrix}0 & i\\ 1 & i\end{matrix}$	$\begin{matrix}i & i\\ 1 & i\end{matrix}$	$\begin{matrix}0 & 0\\ 1+i & i\end{matrix}$	$\begin{matrix}i & 0\\ 1+i & i\end{matrix}$	$\begin{matrix}0 & i\\ 1+i & i\end{matrix}$	$\begin{matrix}i & i\\ 1+i & i\end{matrix}$
$\begin{matrix}1 & 0\\ 1 & 0\end{matrix}$	$\begin{matrix}1+i & 0\\ 1 & 0\end{matrix}$	$\begin{matrix}1 & i\\ 1 & 0\end{matrix}$	$\begin{matrix}1+i & i\\ 1 & 0\end{matrix}$	$\begin{matrix}1 & 0\\ 1+i & 0\end{matrix}$	$\begin{matrix}1+i & 0\\ 1+i & 0\end{matrix}$	$\begin{matrix}1 & i\\ 1+i & 0\end{matrix}$	$\begin{matrix}1+i & i\\ 1+i & 0\end{matrix}$	$\begin{matrix}1 & 0\\ 1 & i\end{matrix}$	$\begin{matrix}1+i & 0\\ 1 & i\end{matrix}$	$\begin{matrix}1 & i\\ 1 & i\end{matrix}$	$\begin{matrix}1+i & i\\ 1 & i\end{matrix}$	$\begin{matrix}1 & 0\\ 1+i & i\end{matrix}$	$\begin{matrix}1+i & 0\\ 1+i & i\end{matrix}$	$\begin{matrix}1 & i\\ 1+i & i\end{matrix}$	$\begin{matrix}1+i & i\\ 1+i & i\end{matrix}$
$\begin{matrix}0 & 1\\ 1 & 0\end{matrix}$	$\begin{matrix}i & 1\\ 1 & 0\end{matrix}$	$\begin{matrix}0 & 1+i\\ 1 & 0\end{matrix}$	$\begin{matrix}i & 1+i\\ 1 & 0\end{matrix}$	$\begin{matrix}0 & 1\\ 1+i & 0\end{matrix}$	$\begin{matrix}i & 1\\ 1+i & 0\end{matrix}$	$\begin{matrix}0 & 1+i\\ 1+i & 0\end{matrix}$	$\begin{matrix}i & 1+i\\ 1+i & 0\end{matrix}$	$\begin{matrix}0 & 1\\ 1 & i\end{matrix}$	$\begin{matrix}i & 1\\ 1 & i\end{matrix}$	$\begin{matrix}0 & 1+i\\ 1 & i\end{matrix}$	$\begin{matrix}i & 1+i\\ 1 & i\end{matrix}$	$\begin{matrix}0 & 1\\ 1+i & i\end{matrix}$	$\begin{matrix}i & 1\\ 1+i & i\end{matrix}$	$\begin{matrix}0 & 1+i\\ 1+i & i\end{matrix}$	$\begin{matrix}i & 1+i\\ 1+i & i\end{matrix}$
$\begin{matrix}1 & 1\\ 1 & 0\end{matrix}$	$\begin{matrix}1+i & 1\\ 1 & 0\end{matrix}$	$\begin{matrix}1 & 1+i\\ 1 & 0\end{matrix}$	$\begin{matrix}1+i & 1+i\\ 1 & 0\end{matrix}$	$\begin{matrix}1 & 1\\ 1+i & 0\end{matrix}$	$\begin{matrix}1+i & 1\\ 1+i & 0\end{matrix}$	$\begin{matrix}1 & 1+i\\ 1+i & 0\end{matrix}$	$\begin{matrix}1+i & 1+i\\ 1+i & 0\end{matrix}$	$\begin{matrix}1 & 1\\ 1 & i\end{matrix}$	$\begin{matrix}1+i & 1\\ 1 & i\end{matrix}$	$\begin{matrix}1 & 1+i\\ 1 & i\end{matrix}$	$\begin{matrix}1+i & 1+i\\ 1 & i\end{matrix}$	$\begin{matrix}1 & 1\\ 1+i & i\end{matrix}$	$\begin{matrix}1+i & 1\\ 1+i & i\end{matrix}$	$\begin{matrix}1 & 1+i\\ 1+i & i\end{matrix}$	$\begin{matrix}1+i & 1+i\\ 1+i & i\end{matrix}$
$\begin{matrix}0 & 0\\ 0 & 1\end{matrix}$	$\begin{matrix}i & 0\\ 0 & 1\end{matrix}$	$\begin{matrix}0 & i\\ 0 & 1\end{matrix}$	$\begin{matrix}i & i\\ 0 & 1\end{matrix}$	$\begin{matrix}0 & 0\\ i & 1\end{matrix}$	$\begin{matrix}i & 0\\ i & 1\end{matrix}$	$\begin{matrix}0 & i\\ i & 1\end{matrix}$	$\begin{matrix}i & i\\ i & 1\end{matrix}$	$\begin{matrix}0 & 0\\ 0 & 1+i\end{matrix}$	$\begin{matrix}i & 0\\ 0 & 1+i\end{matrix}$	$\begin{matrix}0 & i\\ 0 & 1+i\end{matrix}$	$\begin{matrix}i & i\\ 0 & 1+i\end{matrix}$	$\begin{matrix}0 & 0\\ i & 1+i\end{matrix}$	$\begin{matrix}i & 0\\ i & 1+i\end{matrix}$	$\begin{matrix}0 & i\\ i & 1+i\end{matrix}$	$\begin{matrix}i & 0\\ i & 1+i\end{matrix}$
$\begin{matrix}1 & 0\\ 0 & 1\end{matrix}$	$\begin{matrix}1+i & 0\\ 0 & 1\end{matrix}$	$\begin{matrix}1 & i\\ 0 & 1\end{matrix}$	$\begin{matrix}1+i & i\\ 0 & 1\end{matrix}$	$\begin{matrix}1 & 0\\ i & 1\end{matrix}$	$\begin{matrix}1+i & 0\\ i & 1\end{matrix}$	$\begin{matrix}1 & i\\ i & 1\end{matrix}$	$\begin{matrix}1+i & i\\ i & 1\end{matrix}$	$\begin{matrix}1 & 0\\ 0 & 1+i\end{matrix}$	$\begin{matrix}1+i & 0\\ 0 & 1+i\end{matrix}$	$\begin{matrix}1 & i\\ 0 & 1+i\end{matrix}$	$\begin{matrix}1+i & i\\ 0 & 1+i\end{matrix}$	$\begin{matrix}1 & 0\\ i & 1+i\end{matrix}$	$\begin{matrix}1+i & 0\\ i & 1+i\end{matrix}$	$\begin{matrix}1 & i\\ i & 1+i\end{matrix}$	$\begin{matrix}1+i & i\\ i & 1+i\end{matrix}$
$\begin{matrix}0 & 1\\ 0 & 1\end{matrix}$	$\begin{matrix}i & 1\\ 0 & 1\end{matrix}$	$\begin{matrix}0 & 1+i\\ 0 & 1\end{matrix}$	$\begin{matrix}i & 1+i\\ 0 & 1\end{matrix}$	$\begin{matrix}0 & 1\\ i & 1\end{matrix}$	$\begin{matrix}i & 1\\ i & 1\end{matrix}$	$\begin{matrix}0 & 1+i\\ i & 1\end{matrix}$	$\begin{matrix}i & 1+i\\ i & 1\end{matrix}$	$\begin{matrix}0 & 1\\ 0 & 1+i\end{matrix}$	$\begin{matrix}i & 1\\ 0 & 1+i\end{matrix}$	$\begin{matrix}0 & 1+i\\ 0 & 1+i\end{matrix}$	$\begin{matrix}i & 1+i\\ 0 & 1+i\end{matrix}$	$\begin{matrix}0 & 1\\ i & 1+i\end{matrix}$	$\begin{matrix}i & 1\\ i & 1+i\end{matrix}$	$\begin{matrix}0 & 1+i\\ i & 1+i\end{matrix}$	$\begin{matrix}i & 1+i\\ i & 1+i\end{matrix}$
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$\begin{matrix}0 & 0\\ 1 & 1\end{matrix}$	$\begin{matrix}i & 0\\ 1 & 1\end{matrix}$	$\begin{matrix}0 & i\\ 1 & 1\end{matrix}$	$\begin{matrix}i & i\\ 1 & 1\end{matrix}$	$\begin{matrix}0 & 0\\ 1+i & 1\end{matrix}$	$\begin{matrix}i & 0\\ 1+i & 1\end{matrix}$	$\begin{matrix}0 & i\\ 1+i & 1\end{matrix}$	$\begin{matrix}i & i\\ 1+i & 1\end{matrix}$	$\begin{matrix}0 & 0\\ 1 & 1+i\end{matrix}$	$\begin{matrix}i & 0\\ 1 & 1+i\end{matrix}$	$\begin{matrix}0 & i\\ 1 & 1+i\end{matrix}$	$\begin{matrix}i & i\\ 1 & 1+i\end{matrix}$	$\begin{matrix}0 & 0\\ 1+i & 1+i\end{matrix}$	$\begin{matrix}i & 0\\ 1+i & 1+i\end{matrix}$	$\begin{matrix}0 & i\\ 1+i & 1+i\end{matrix}$	$\begin{matrix}i & i\\ 1+i & 1+i\end{matrix}$
$\begin{matrix}1 & 0\\ 1 & 1\end{matrix}$	$\begin{matrix}1+i & 0\\ 1 & 1\end{matrix}$	$\begin{matrix}1 & i\\ 1 & 1\end{matrix}$	$\begin{matrix}1+i & i\\ 1 & 1\end{matrix}$	$\begin{matrix}1 & 0\\ 1+i & 1\end{matrix}$	$\begin{matrix}1+i & 0\\ 1+i & 1\end{matrix}$	$\begin{matrix}1 & i\\ 1+i & 1\end{matrix}$	$\begin{matrix}1+i & i\\ 1+i & 1\end{matrix}$	$\begin{matrix}1 & 0\\ 1 & 1+i\end{matrix}$	$\begin{matrix}1+i & 0\\ 1 & 1+i\end{matrix}$	$\begin{matrix}1 & i\\ 1 & 1+i\end{matrix}$	$\begin{matrix}1+i & i\\ 1 & 1+i\end{matrix}$	$\begin{matrix}1 & 0\\ 1+i & 1+i\end{matrix}$	$\begin{matrix}1+i & 0\\ 1+i & 1+i\end{matrix}$	$\begin{matrix}1 & i\\ 1+i & 1+i\end{matrix}$	$\begin{matrix}1+i & i\\ 1+i & 1+i\end{matrix}$
$\begin{matrix}0 & 1\\ 1 & 1\end{matrix}$	$\begin{matrix}i & 1\\ 1 & 1\end{matrix}$	$\begin{matrix}0 & 1+i\\ 1 & 1\end{matrix}$	$\begin{matrix}i & 1+i\\ 1 & 1\end{matrix}$	$\begin{matrix}0 & 1\\ 1+i & 1\end{matrix}$	$\begin{matrix}i & 1\\ 1+i & 1\end{matrix}$	$\begin{matrix}0 & 1+i\\ 1+i & 1\end{matrix}$	$\begin{matrix}i & 1+i\\ 1+i & 1\end{matrix}$	$\begin{matrix}0 & 1\\ 1 & 1+i\end{matrix}$	$\begin{matrix}i & 1\\ 1 & 1+i\end{matrix}$	$\begin{matrix}0 & 1+i\\ 1 & 1+i\end{matrix}$	$\begin{matrix}i & 1+i\\ 1 & 1+i\end{matrix}$	$\begin{matrix}0 & 1\\ 1+i & 1+i\end{matrix}$	$\begin{matrix}i & 1\\ 1+i & 1+i\end{matrix}$	$\begin{matrix}0 & 1+i\\ 1+i & 1+i\end{matrix}$	$\begin{matrix}i & 1+i\\ 1+i & 1+i\end{matrix}$
$\begin{matrix}1 & 1\\ 1 & 1\end{matrix}$	$\begin{matrix}1+i & 1\\ 1 & 1\end{matrix}$	$\begin{matrix}1 & 1+i\\ 1 & 1\end{matrix}$	$\begin{matrix}1+i & 1+i\\ 1 & 1\end{matrix}$	$\begin{matrix}1 & 1\\ 1+i & 1\end{matrix}$	$\begin{matrix}1+i & 1\\ 1+i & 1\end{matrix}$	$\begin{matrix}1 & 1+i\\ 1+i & 1\end{matrix}$	$\begin{matrix}1+i & 1+i\\ 1+i & 1\end{matrix}$	$\begin{matrix}1 & 1\\ 1 & 1+i\end{matrix}$	$\begin{matrix}1+i & 1\\ 1 & 1+i\end{matrix}$	$\begin{matrix}1 & 1+i\\ 1 & 1+i\end{matrix}$	$\begin{matrix}1+i & 1+i\\ 1 & 1+i\end{matrix}$	$\begin{matrix}1 & 1\\ 1+i & 1+i\end{matrix}$	$\begin{matrix}1+i & 1\\ 1+i & 1+i\end{matrix}$	$\begin{matrix}1 & 1+i\\ 1+i & 1+i\end{matrix}$	$\begin{matrix}1+i & 1+i\\ 1+i & 1+i\end{matrix}$

If the elements of the characteristics are conjugate imaginaries, that is, if we have $\alpha_r + i\beta_r = a_r - ib_r$, the total number of Ξ-functions for any given number of arguments is equal to the total number of theta-functions for the same number of arguments, *i. e.* in the case of p-tuple functions if the characteristic is

$$\begin{pmatrix} a_1 + ib_1, & a_2 + ib_2 \dots & a_p + ib_p \\ a_1 - ib_1, & a_2 - ib_2 \dots & a_p - ib_p \end{pmatrix}$$

(where a, b take the values zero and unity), we have 4^p Ξ-functions. For $p = 1$ the characteristics are

5. $$\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} i \\ -i \end{pmatrix}, \begin{pmatrix} 1+i \\ 1-i \end{pmatrix};$$

for $p = 2$ they are

6. $$\begin{array}{cccc} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, & \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, & \begin{pmatrix} 0 & i \\ 0 & -i \end{pmatrix}, & \begin{pmatrix} 0 & 1+i \\ 0 & 1-i \end{pmatrix} \\ \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, & \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, & \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}, & \begin{pmatrix} 1 & 1+i \\ 1 & 1-i \end{pmatrix} \\ \begin{pmatrix} i & 0 \\ -i & 0 \end{pmatrix}, & \begin{pmatrix} i & 1 \\ -i & 1 \end{pmatrix}, & \begin{pmatrix} i & i \\ -i & -i \end{pmatrix}, & \begin{pmatrix} i & 1+i \\ -i & 1-i \end{pmatrix} \\ \begin{pmatrix} 1+i & 0 \\ 1-i & 0 \end{pmatrix}, & \begin{pmatrix} 1+i & 1 \\ 1-i & 1 \end{pmatrix}, & \begin{pmatrix} 1+i & i \\ 1-i & -i \end{pmatrix}, & \begin{pmatrix} 1+i & 1+i \\ 1-i & 1-i \end{pmatrix}. \end{array}$$

If in the function $$\Xi\begin{pmatrix} l_r \\ \lambda_r \end{pmatrix}(v_r), \quad r = 1, 2, \dots p,$$

we change l_r into $l_r + 2y_r$, where y_r is an integer, there will obviously be no change in the value of the function. If, however, we change λ_r into $\lambda_r + 2x_r$, where x_r is again an integer, we shall have as the new linear term in the exponential $\tfrac{1}{2}\pi i \{(n_1 + l_1)(v_1 + \lambda_1 + 2x_1) + \dots + (n_p + l_p)(v_p + \lambda_p + 2x_p)\}$, which differs from the original term by the quantity

$$\pi i \Sigma n_r x_r + \pi i \Sigma l_r x_r, \quad r = 1, 2, \dots p.$$

The term $\pi i \Sigma n_r x_r$ is, since n is even, an even multiple of πi and so produces no change in the value of the function; the second term $\pi i \Sigma l_r x_r$ causes the function to be multiplied by the factor $(-)^{\Sigma l_r x_r}$, that is, we have

7. $$\Xi\begin{pmatrix} l_r + 2y_r \\ \lambda_r + 2x_r \end{pmatrix}(v_r) = (-)^{\Sigma l_r x_r} \Xi\begin{pmatrix} l_r \\ \lambda_r \end{pmatrix}(v_r);$$

since $l_r = a_r + ib_r =$, say, $A_r e^{i\phi_r}$, this may be written

8. $$\Xi\begin{pmatrix} l_r + 2y_r \\ \lambda_r + 2x_r \end{pmatrix}(v_r) = (-)^{\Sigma x_r A_r e^{i\phi_r}} . \Xi\begin{pmatrix} l_r \\ \lambda_r \end{pmatrix}(v_r).$$

On the assumption that a and b are each either zero or unity the modulus of l is either 0, 1, or $\sqrt{2}$, so that in the sum $\Sigma x_r A_r e^{i\phi_r}$ each term is either 0, $xe^{i\phi}$ or $x\sqrt{2}e^{i\phi}$, and therefore the factor $(-)^{\Sigma x_r A_r e^{i\phi_r}}$ in the preceding equation takes in all 2^{2p} forms; these are not all distinct and only one of them makes the factor

$= 1$, viz. when all the moduli A are $= 0$. Since the four possible moduli of l are $A = 0, 1, 1, \sqrt{2}$, it would appear that the total number of independent values taken by $\Sigma x_r A_r e^{i\phi_r}$ is equal to 3^p. Among these 3^p quantities a certain number would seem to be repeated two or more (always an even number) of times to make up the 2^{2p} values of $\Sigma x_r A_r e^{i\phi_r}$. It is easy to see that the total number of values of this quantity which do not seem to be repeated is equal to $3^{p-2}.2^2$ (for $p \geqq 2$), and so the total number of values which are repeated is

$$3^p - 3^{p-2}.2^2 = 3^{p-2}.(3^2 - 2^2) = 5.3^{p-2}.$$

The moduli having the above values it is easy to see what the corresponding arguments are, and so to show that the above conclusions do not hold. Take the element $l, = a + ib$. This may also be written $l = Ae^{i\phi}$; suppose now that $a = 0$ and $b = 0$, then $A = 0$ and ϕ is indeterminate, but since in this case the term $xAe^{i\phi}$ vanishes, it is unnecessary to consider ϕ at all. Suppose next that $b = 0$ and $a = 1$, then we have at once $\phi = 0$, or 2π; now let $a = 0$ and $b = 1$ and we have $\phi = \frac{\pi}{2}$; finally if $a = 1$ and $b = 1$ we find $\phi = \frac{\pi}{4}$.

Take as illustration the case of the double Ξ-functions where the characteristic is

$$\begin{pmatrix} l_1 & l_2 \\ \lambda_1 & \lambda_2 \end{pmatrix} \equiv \begin{pmatrix} a_1 + ib_1 & a_2 + ib_2 \\ \alpha_1 + i\beta_1 & \alpha_2 + i\beta_2 \end{pmatrix};$$

the quantity to be examined is

$$(-)^{x_1 A_1 e^{i\phi_1} + x_2 A_2 e^{i\phi_2}}, \text{ say } (-)^R;$$

now R takes 16 values, these are

$$\left.\begin{matrix} a_1 = 0, & b_1 = 0 \\ a_2 = 0, & b_2 = 0 \end{matrix}\right\} 0 \qquad \left.\begin{matrix} a_1 = 0, & b_1 = 0 \\ a_2 = 1, & b_2 = 0 \end{matrix}\right\} x_2 . 1 . e^{i0}$$

$$\left.\begin{matrix} a_1 = 0, & b_1 = 0 \\ a_2 = 0, & b_2 = 1 \end{matrix}\right\} x_2 . 1 . e^{i\frac{\pi}{2}} \qquad \left.\begin{matrix} a_1 = 0, & b_1 = 0 \\ a_2 = 1, & b_2 = 1 \end{matrix}\right\} x_2 \sqrt{2} . e^{i\frac{\pi}{4}}$$

$$\left.\begin{matrix} a_1 = 1, & b_1 = 0 \\ a_2 = 0, & b_2 = 0 \end{matrix}\right\} x_1 . 1 . e^{i0} \qquad \left.\begin{matrix} a_1 = 1, & b_1 = 0 \\ a_2 = 1, & b_2 = 0 \end{matrix}\right\} x_1 . 1 . e^{i0} + x_2 . 1 . e^{i0}$$

$$\left.\begin{matrix} a_1 = 1, & b_1 = 0 \\ a_2 = 0, & b_1 = 1 \end{matrix}\right\} x_1 . 1 . e^{i0} + x_2 . 1 . e^{i\frac{\pi}{2}} \qquad \left.\begin{matrix} a_1 = 1, & b_1 = 0 \\ a_2 = 1, & b_2 = 1 \end{matrix}\right\} x_1 . 1 . e^{i0} + x_2 \sqrt{2} . e^{i\frac{\pi}{4}}$$

9. $$\left.\begin{matrix} a_1 = 0, & b_1 = 1 \\ a_2 = 0, & b_2 = 0 \end{matrix}\right\} x_1 . 1 . e^{i\frac{\pi}{2}} \qquad \left.\begin{matrix} a_1 = 0, & b_1 = 1 \\ a_2 = 1, & b_2 = 0 \end{matrix}\right\} x_1 . 1 . e^{i\frac{\pi}{2}} + x_2 . 1 . e^{i0}$$

$$\left.\begin{matrix} a_1 = 0, & b_1 = 1 \\ a_2 = 0, & b_2 = 1 \end{matrix}\right\} x_1 . 1 . e^{i\frac{\pi}{2}} + x_2 . 1 . e^{i\frac{\pi}{2}} \qquad \left.\begin{matrix} a_1 = 0, & b_1 = 1 \\ a_2 = 1, & b_2 = 1 \end{matrix}\right\} x_1 . 1 . e^{i\frac{\pi}{2}} + x_2 \sqrt{2} . e^{i\frac{\pi}{4}}$$

$$\left.\begin{matrix} a_1 = 1, & b_1 = 1 \\ a_2 = 0, & b_2 = 0 \end{matrix}\right\} x_1 \sqrt{2} . e^{i\frac{\pi}{4}} \qquad \left.\begin{matrix} a_1 = 1, & b_1 = 1 \\ a_2 = 1, & b_2 = 0 \end{matrix}\right\} x_1 \sqrt{2} . e^{i\frac{\pi}{4}} + x_2 . 1 . e^{i0}$$

$$\left.\begin{matrix} a_1 = 1, & b_1 = 1 \\ a_2 = 0, & b_2 = 0 \end{matrix}\right\} x_1 \sqrt{2} . e^{i\frac{\pi}{4}} + x_2 . 1 . e^{i\frac{\pi}{2}} \qquad \left.\begin{matrix} a_1 = 1, & b_1 = 1 \\ a_2 = 1, & b_2 = 1 \end{matrix}\right\} x_1 \sqrt{2} . e^{i\frac{\pi}{4}} + x_2 \sqrt{2} . e^{i\frac{\pi}{4}}$$

or,

$$10.\quad R=\left\{\begin{matrix} 0, & x_2, & ix_2, & x_2(1+i), \\ x_1, & x_1+x_2, & x_1+ix_2, & x_1+x_2(1+i), \\ ix_1, & ix_1+x_2, & ix_1+ix_2, & ix_1+x_2(1+i), \\ x_1(1+i), & x_1(1+i)+x_2, & x_1(1+i)+ix_2, & x_1(1+i)+x_2(1+i)\cdot \end{matrix}\right\}$$

The entire 16 values of R are all different, the one from the other, and also, in the general case, the entire 2^{2p} values are all different, one from the other. From these values of R we have, since $(-)^i = e^{-\pi}$,

$$11.\ (-)^R=\left\{\begin{matrix} 1, & (-)^{x_2}, & e^{-x_2\pi}, & (-)^{x_2}.e^{-x_2\pi}, \\ (-)^{x_1}, & (-)^{x_1+x_2}, & (-)^{x_1}.e^{-x_2\pi}, & (-)^{(x_1+x_2)}.e^{-x_2\pi}, \\ e^{-x_1\pi}, & (-)^{x_2}.e^{-x_1\pi}, & e^{-(x_1+x_2)\pi}, & (-)^{x_2}.e^{-(x_1+x_2)\pi}, \\ (-)^{x_1}.e^{-x_1\pi}, & (-)^{(x_1+x_2)}.e^{-x_1\pi}, & (-)^{x_1}.e^{-(x_1+x_2)\pi}, & (-)^{(x_1+x_2)}.e^{-(x_1+x_2)\pi}. \end{matrix}\right\}$$

From this we see that the factor $(-)^{\Sigma x_r A_r e^{i\phi_r}}$ is always either $+1$, -1, or $\pm e^{-k\pi}$, where k is an integer. In the case of the double Ξ-functions there are thus 16 functions which only change sign by replacing in the characteristic λ by $\lambda + 2x$, and so in the p-tuple Ξ-functions there will be 2^{2p} functions which will at the most only change sign by replacing λ by $\lambda + 2x$. These functions of course correspond to those values of the characteristics in which all of the b's are made zero and the a's take the values zero and unity. When this happens the Ξ-functions become the ordinary theta-functions. As the equation

$$12.\qquad \Xi\begin{pmatrix} l_r + 2y_r \\ \lambda_r + 2x_r \end{pmatrix}(v_r) = (-)^{\Sigma x_r A_r e^{i\phi_r}}.\Xi\begin{pmatrix} l_r \\ \lambda_r \end{pmatrix}(v_r)$$

can, by virtue of the factor $(-)^{\Sigma x_r A_r e^{i\phi_r}}$ only take 2^{3p} different forms, and as in 2^{2p} of these the factor becomes either $+1$ or -1 there remain

$$2^{3p} - 2^{2p} = 2^{2p}(2^p - 1)$$

cases in which the factor becomes $= \pm e^{-k\pi}$, a constant whose greatest value for k positive is $e^{-\pi}$, or $(-)^i$.

For the theta-functions the quantities l and λ reduce to their real parts a and α, and further, a and α are integers. Observing this fact and also the fact that the summations with respect to $n_1, n_2, n_3 \ldots$ extend from $-\infty$ to $+\infty$, we readily find the number of even and odd theta-functions, that is, the number of functions which are not altered by changing the arguments $v_1, v_2 \ldots v_p$ into $-v_1, -v_2 \ldots -v_p$. The determination of the number of odd and even theta-functions is made to depend upon the fact that in the exponent of the general term of the series we can change $n + l$ into $-n - l$, where of course l is a real integer. In the present case such a change obviously cannot be made, since l is a complex quantity, viz. $l = a + ib$. It seems as if a formula of transformation ought to be found which would represent the result of changing v into $-v$, and which would reduce to the corresponding formula for theta-functions by making b and β each zero, but I have not been able up to the present to hit upon it. I have found two methods of transforming the Ξ-functions, so that when the elements of the characteristic reduce to real quantities (integers) the corresponding theta-function formulæ are obtained. I shall only give one of them in what follows, viz. the one which, on making the b's and β's all zero, will give all of the corresponding formulæ for the theta-functions. The Ξ-function is defined by the equation

$$13. \qquad \Xi\begin{pmatrix} l_r \\ \lambda_r \end{pmatrix}(v_r) = \sum_{n=-\infty}^{n=+\infty} \exp.\begin{pmatrix} n_r + l_r \\ v_r + \lambda_r \end{pmatrix}, \; r = 1, 2, 3 \ldots$$

The n's are of course all even, and a single summation sign is used simply for brevity. As above we have $l = a + ib$ and $\lambda = \alpha + i\beta$, with the above restriction (which is at first not necessary) that a, b, α and β take only the values zero and unity. Now remembering that

$$\exp.\begin{pmatrix} n_r + l_r \\ v_r + \lambda_r \end{pmatrix} = \exp.\Big\{\frac{1}{4}(*\between n_1 + l_1, n_2 + l_2 \ldots n_p + l_p)^2 + \frac{1}{2}\pi i \sum_{r=1}^{r=p} (n_r + l_r)(v_r + \lambda_r)\Big\},$$

and remembering that the summation with respect to n is (for even integers) from $-\infty$ to $+\infty$, we can change n into $-n$ without altering the value of the function. We now seek the value of the function

$$\Xi\begin{pmatrix} (-1 + ib_r)l_r \\ (1 + i\beta_r)\lambda_r \end{pmatrix}(-v_r), \; r = 1, 2, \ldots p.$$

Changing n into $-n$ the exponent of the general term becomes

$$\tfrac{1}{4}(*\between -n_1 + (-1 + ib_1)l_1, \ldots -n_p + (-1 + ib_p)l_p)^2$$
$$+ \tfrac{1}{2}\pi i\{[-n_1 + (-1 + ib_1)l_1][-v_1 + (1 + i\beta_1)\lambda_1] + \ldots$$
$$+ [-n_p + (-1 + ib_p)l_p][-v_p + (1 + i\beta_p)\lambda_p]\},$$

or

$$\tfrac{1}{4}(*\between n_1 + (1 - ib_1)l_1, \ldots n_p + (1 - ib_p)l_p)^2$$
$$+ \tfrac{1}{2}\pi i\{[n_1 + (1 - ib_1)l_1][v_1 - (1 + i\beta_1)\lambda_1] + \ldots$$
$$+ [n_p + (1 - ib_p)\lambda_p][v_p - (1 + i\beta_p)\lambda_p]\}.$$

Expanding this and subtracting from the original exponent

$$\frac{1}{4}(*\between n_1 + l_1, \ldots n_p + l_p)^2 + \frac{1}{2}\pi i \sum_{r=1}^{r=p} (n_r + l_r)(v_r + \lambda_r),$$

we have, after some simple reductions,

14. $$\tfrac{1}{4}\{-\Sigma l_r^2 - 2i\Sigma n_r b_r + \Sigma l_r^2(1-ib_r)^2 + i\Sigma\Sigma l_r b_r(n_s+l_s) - \Sigma\Sigma l_r l_s b_r b_s\} \\ + \tfrac{1}{2}\pi i\{\Sigma(2+i\beta_r)n_r\lambda_r + i\Sigma b_r l_r v_r + \Sigma(2+i\beta_r - ib_r + b_r\beta_r)l_r\lambda_r\},$$

the summations extending from $r=1$ to $r=p$ and in the double summations the values $r=s$ being excluded. Write this as

15. $$\tfrac{1}{4}\Phi_r + \tfrac{1}{2}\pi i\Psi_{ri};$$

if now we make $b=\beta=0$ we have

$$\Phi_r = 0 \text{ and } \Psi_r = 2\Sigma n_r\alpha_r + 2\Sigma a_r\alpha_r,$$

which is the corresponding quantity for the ϑ-functions. We have now for the Ξ-functions the relation

16. $$\Xi\begin{pmatrix}(-1+ib_r)l_r\\(1+i\beta_r)\lambda_r\end{pmatrix}(-v_r) = \Xi\begin{pmatrix}l_r\\\lambda_r\end{pmatrix}(v_r).\exp.\{\tfrac{1}{4}\Phi_r + \tfrac{1}{2}\pi i\Psi_r\}.$$

Making $b=\beta=0$ in this and we have, replacing Ξ by Θ, the well-known formula

17. $$\Theta\begin{pmatrix}a_1a_2\ldots a_p\\\alpha_1\alpha_2\ldots\alpha_p\end{pmatrix}(-v_1, -v_2 \ldots -v_p) = (-)^{\Sigma a_r\alpha_r}\Theta\begin{pmatrix}a_1a_2\ldots a_p\\\alpha_1\alpha_2\ldots\alpha_p\end{pmatrix}(v_1, v_2\ldots v_p).$$

Another and rather more general form is obtained by changing l into $(-1+ib)l$, λ into $(1+i\beta)\lambda$ and v into $(-1+i\beta)v$. The quantity called Φ_r above undergoes no change, but we have, calling Ψ'_r the new value of Ψ_r

18. $$\Psi'_r = i\Sigma\beta_r n_r v_r + \Sigma(2+i\beta_r)n_r\lambda_r + \Sigma(ib_r + i\beta_r + b_r\beta_r)l_r v_r \\ + \Sigma(2+i\beta_r - ib_r + b_r\beta_r)l_r\lambda_r,$$

and

19. $$\Psi'_r - \Psi_r = i\Sigma\beta_r n_r v_r + \Sigma(i\beta_r + b_r\beta_r)l_r v_r + 2i\Sigma b_r l_r\lambda_r,$$

every term of which vanishes when l and λ are real. The quantity $\tfrac{1}{4}\Phi_r$ may be written

20. $$\tfrac{1}{4}\Phi_r = -\tfrac{i}{2}\Sigma n_r b_r - \tfrac{i}{2}\Sigma b_r l_r^2 - \tfrac{1}{4}\Sigma l_r^2 b_r^2 + \tfrac{i}{4}\Sigma\Sigma l_r b_r(n_s+l_s) - \tfrac{1}{4}\Sigma\Sigma l_r l_s b_r b_s;$$

since n is an even integer write $n=2k$, then this is

21. $$= -i\Sigma k_r b_r - \tfrac{1}{2}i\Sigma b_r l_r^2 - \tfrac{1}{4}\Sigma l_r^2 b_r^2 + \tfrac{i}{4}\Sigma\Sigma l_r b_r(2k_s + l_s) - \tfrac{1}{4}\Sigma\Sigma l_r l_s b_r b_s;$$

again since $b=0$ or 1 the first term of this becomes a real multiple of i, and the remaining terms become quantities of the form $M+iN$, where M and N are real and of the forms μ, $\frac{\nu}{2}$, $\frac{\rho}{4}$, where μ, ν and ρ are real integers. Similar remarks might be made concerning Ψ_r, but, for the present, it is not necessary to consider this quantity We may make one further transformation similar to the above and to the corresponding transformation in the case of the theta-functions. For simplicity take the function

22. $$\Xi\begin{pmatrix}a_1+ib_1, & a_2+ib_2\\\alpha_1+i\beta_1, & \alpha_2+i\beta_2\end{pmatrix}(v_1, v_2) = \Sigma\Sigma\exp.[\tfrac{1}{4}(a_{11}, a_{12}, a_{22}\between n_1+a_1+ib_1, n_2+a_2+ib_2)^2 \\ + \tfrac{1}{2}\pi i\{(n_1+a_1+ib_1)(v_1+\alpha_1+i\beta_1) + (n_2+a_2+ib_2)(v_2+\alpha_2+i\beta_2)\}].$$

Now we can change n_1 and n_2 into $-n_1$ and $-n_2$ without altering the value of the function; make this change and also change l_1 and l_2 into $-l_1$ and $-l_2$ and v_1 and v_2 into $-v_1$ and $-v_2$. We have then to find the value of the function,

$$\Xi\begin{pmatrix} -l_1 & -l_2 \\ \lambda_1 & \lambda_2 \end{pmatrix}(-v_1, -v_2).$$

The exponent is, on changing n into $-n$,

$$\tfrac{1}{4}(*)(-n_1-l_1, -n_2-l_2)^2+\tfrac{1}{2}\pi i\{(-n_1-l_1)(-v_1+\lambda_1)+(-n_2-l_2)(-v_2+\lambda_2)\}.$$

The quadratic term is unaltered, but the linear term becomes

$$\tfrac{1}{2}\pi i\{(n_1+l_1)(v_1+\lambda_1)+(n_2+l_2)(v_2+\lambda_2)\}-\pi i\{(n_1+l_1)\lambda_1+(n_2+l_2)\lambda_2\};$$

this differs from the original exponent by the term,

23. $$-\pi i(n_1\lambda_1+n_2\lambda_2)-\pi i(l_1\lambda_1+l_2\lambda_2),$$

say,

23′. $$-\pi iN-\pi iL.$$

Make now $n=2k$ and give λ_1 and λ_2 all their values: these are

$$\lambda_1=0,\ 1,\ i,\ 1+i,$$
$$\lambda_2=0,\ 1,\ i,\ 1+i;$$

then all possible values of N are

24. $$\begin{array}{cccc} 0, & 2k_2, & 2ik_2, & 2k_2+2ik_2, \\ 2k_1, & 2k_1+2k_2, & 2k_1+2ik_2, & 2k_1+2k_2+2ik_2, \\ 2ik_1, & 2ik_1+2k_2, & 2ik_1+2ik_2, & 2ik_1+2ik_2+2k_2, \\ 2k_1+2ik_1, & 2k_1+2k_2+2ik_2, & 2k_1+2ik_1+2ik_2, & 2k_1+2k_2+2ik_1+2ik_2. \end{array}$$

(Notice in forming this table that after writing down the first line and first column all the remaining terms are formed by combining them: viz., the second, third and fourth elements in the second column are obtained by adding the first element in this column to the second, third and fourth elements in the first column respectively, etc.). The exponential thus takes one of the values,

25. $$+1,\ e^{2\tau\pi},$$

where τ is a positive or negative integer, when λ reduces to its real part 0 or 1 this exponential is $=+1$. There are in all 256 different values of L, 16 of which for l and $\lambda=0,\ 1$, make the exponential $e^{-\pi i(l_1\lambda_1+l_2\lambda_2)}$ equal to ± 1, ten values giving $+1$ and six values giving -1. We have finally

26. $$\Xi\begin{pmatrix} -l_1 & -l_2 \\ \lambda_1 & \lambda_2 \end{pmatrix}(-v_1, -v_2)$$
$$=(-)^{l_1\lambda_1+l_2\lambda_2}\Sigma\Sigma(1)^{k_1l_1+k_2l_2}\text{ exp. }\{\tfrac{1}{4}(a_{11}, a_{12}, a_{22})(n_1+l_1, n_2+l_2)^2$$
$$+\tfrac{1}{2}\pi i[(n_1+l_1)(v_1+\lambda_1)+(n_2+l_2)(v_2+\lambda_2)]\}.$$

For the general case we have obviously

$$27.\qquad \Xi\begin{pmatrix} -l_1 \dots -l_p \\ \lambda_1 \dots \lambda_p \end{pmatrix}(-v_1, \dots, -v_p) = (-)^{\Sigma l_r\lambda_r}\Sigma \dots \Sigma(1)^{\Sigma k_r l_r}$$

$$\times \exp. \{\tfrac{1}{4}(*\between n_1 + l_1, \dots n_p + l_p)^2 + \tfrac{1}{2}\pi i\Sigma(n_r + l_r)(v_r + \lambda_r)\}.$$

For $p = 2$ write $l_1\lambda_1 + l_2\lambda_2 = \sigma$; the entire set of values of σ are given in the following table:

28.

<table>
<tr><td>0</td><td>0</td><td>0</td><td>0</td><td colspan="4" rowspan="4">A</td><td colspan="4" rowspan="4">A</td><td colspan="4" rowspan="4">A</td></tr>
<tr><td>0</td><td>1</td><td>i</td><td>1+i</td></tr>
<tr><td>0</td><td>i</td><td>−1</td><td>−1+i</td></tr>
<tr><td>0</td><td>1+i</td><td>−1+i</td><td>2i</td></tr>
<tr><td colspan="4" rowspan="4">A</td><td>1</td><td>1</td><td>1</td><td>1</td><td>i</td><td>i</td><td>i</td><td>i</td><td>1+i</td><td>1+i</td><td>1+i</td><td>1+i</td></tr>
<tr><td>1</td><td>2</td><td>1+i</td><td>2+i</td><td>i</td><td>1+i</td><td>2i</td><td>1+2i</td><td>1+i</td><td>2+i</td><td>1+2i</td><td>2+2i</td></tr>
<tr><td>1</td><td>1+i</td><td>0</td><td>i</td><td>i</td><td>2i</td><td>−1+i</td><td>−1+2i</td><td>1+i</td><td>1+2i</td><td>i</td><td>2i</td></tr>
<tr><td>1</td><td>2+i</td><td>i</td><td>1+2i</td><td>i</td><td>1+2i</td><td>−1+2i</td><td>3i</td><td>1+i</td><td>2+2i</td><td>2i</td><td>1+3i</td></tr>
<tr><td colspan="4" rowspan="4">A</td><td>i</td><td>i</td><td>i</td><td>i</td><td>−1</td><td>−1</td><td>−1</td><td>−1</td><td>−1+i</td><td>−1+i</td><td>−1+i</td><td>−1+i</td></tr>
<tr><td>i</td><td>1+i</td><td>2i</td><td>1+2i</td><td>−1</td><td>0</td><td>−1+i</td><td>i</td><td>−1+i</td><td>i</td><td>−1+2i</td><td>2i</td></tr>
<tr><td>i</td><td>2i</td><td>−1+i</td><td>−1+2i</td><td>−1</td><td>−1+i</td><td>−2</td><td>−2+i</td><td>−1+i</td><td>−1+2i</td><td>−2+i</td><td>−2+2i</td></tr>
<tr><td>i</td><td>1+2i</td><td>−1+2i</td><td>3i</td><td>−1</td><td>i</td><td>−2+i</td><td>−1+2i</td><td>−1+i</td><td>2i</td><td>−2+2i</td><td>−1+3i</td></tr>
<tr><td colspan="4" rowspan="4">A</td><td>1+i</td><td>1+i</td><td>1+i</td><td>1+i</td><td>−1+i</td><td>−1+i</td><td>−1+i</td><td>−1+i</td><td>2i</td><td>2i</td><td>2i</td><td>2i</td></tr>
<tr><td>1+i</td><td>2+i</td><td>1+2i</td><td>2+2i</td><td>−1+i</td><td>i</td><td>−1+2i</td><td>2i</td><td>2i</td><td>1+2i</td><td>3i</td><td>1+3i</td></tr>
<tr><td>1+i</td><td>1+2i</td><td>i</td><td>2i</td><td>−1+i</td><td>−1+2i</td><td>−2+i</td><td>−2+2i</td><td>2i</td><td>3i</td><td>−1+2i</td><td>−1+3i</td></tr>
<tr><td>1+i</td><td>2+2i</td><td>2i</td><td>1+3i</td><td>−1+i</td><td>2i</td><td>−2+2i</td><td>−1+3i</td><td>2i</td><td>1+3i</td><td>−1+3i</td><td>4i</td></tr>
</table>

The spaces marked *A* are to be filled up in the same way as the upper left-hand corner space (each of the sixteen spaces is seen to be symmetrically arranged, and also the entire group is seen to be symmetrical). There are in all only 19 *different* values of σ, viz.

$$0,\ 1,\ 2,\ -1,\ -2,\ i,\ 1+i,\ -1+i,\ 2i,\ 2+i,\ -2+i,\ 1+2i,$$
$$-1+2i,\ 3i,\ 2+2i,\ 1+3i,\ -2+2i,\ -1+3i,\ 4i.$$

The number of times that each of these appears is:

0	number of times		51	$-2+i$	number of times		4
1	"	"	14	$1+2i$	"	"	10
2	"	"	1	$-1+2i$	"	"	10
-1	"	"	14	$3i$	"	"	4
-2	"	"	1	$2+2i$	"	"	4
i	"	"	36	$1+3i$	"	"	4
$1+i$	"	"	32	$-2+2i$	"	"	4
$-1+i$	"	"	32	$-1+3i$	"	"	4
$2i$	"	"	26	$4i$	"	"	1
$2+i$	"	"	4				

making the proper total of 256. The above formula of transformation is more convenient when written in the form,

$$29. \qquad \Xi\begin{pmatrix} -l_1 \dots -l_p \\ \lambda_1 \dots \lambda_p \end{pmatrix}(-v_1, \dots -v_p) = (-)^{\Sigma l_r \lambda_r}\Sigma \dots \Sigma(-)^{\Sigma n_r l_r}$$
$$\times \text{ exp. }\left\{\tfrac{1}{4}(*)(n_1+l_1, \dots n_p+l_p)^2 + \tfrac{1}{2}\pi i\Sigma(n_r+l_r)(v_r+\lambda_r)\right\}.$$

The values (for $p=2$) of $n_1 l_1 + n_2 l_2$ are of course obtained by replacing k_1 and k_2 in (24) by n_1 and n_2. We have then for $n_1 l_1 + n_2 l_2$ the table:

30.

$l_1=0,\ l_2=0$ 0	$l_1=1,\ l_2=0$ n_1	$l_1=i,\ l_2=0$ in_1	$l_1=1+i,\ l_2=0$ n_1+in_1
$l_1=0,\ l_2=1$ n_2	$l_1=1,\ l_2=1$ n_1+n_2	$l_1=i,\ l_2=1$ n_1+in_2	$l_1=1+i,\ l_2=1$ $n_1+n_2+in_2$
$l_1=0,\ l_2=i$ in_2	$l_1=1,\ l_2=i$ n_1+in_2	$l_1=i,\ l_2=i$ in_1+in_2	$l_1=1+i,\ l_2=i$ $in_1+in_2+in_2$
$l_1=0,\ l_2=1+i$ n_2+in_2	$l_1=1,\ l_2=1+i$ $n_1+n_2+in_2$	$l_1=i,\ l_2=1+i$ $in_1+in_2+n_2$	$l_1=1+i,\ l_2=1+i$ $n_1+n_2+in_1+in_2$

The upper line in each block gives the values of l_1 and l_2, and the lower line the corresponding value of $n_1 l_1 + n_2 l_2$.

For greater convenience, table (28) may be presented in the following form. Each square contains all of the values of $l_1\lambda_1 + l_2\lambda_2$ corresponding to the values of l_1 and l_2 written above the square.

31.

$l_1=0,$		$l_2=0$		$l_1=1,$		$l_2=0$		$l_1=i,$		$l_2=0$		$l_1=1+i,$		$l_2=0$	
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	1	1	1	1	i	i	i	i	$1+i$	$1+i$	$1+i$	$1+i$
0	0	0	0	i	i	i	i	-1	-1	-1	-1	$-1+i$	$-1+i$	$-1+i$	$-1+i$
0	0	0	0	$1+i$	$1+i$	$1+i$	$1+i$	$-1+i$	$-1+i$	$-1+i$	$-1+i$	$2i$	$2i$	$2i$	$2i$
$l_1=0,$		$l_2=1$		$l_1=1,$		$l_2=1$		$l_1=i,$		$l_2=1$		$l_1=1+i,$		$l_2=1$	
0	1	i	$1+i$	0	1	i	$1+i$	0	1	i	$1+i$	0	1	i	$1+i$
0	1	i	$1+i$	1	2	$1+i$	$2+i$	i	$1+i$	$2i$	$1+2i$	$1+i$	$2+i$	$1+2i$	$2+2i$
0	1	i	$1+i$	i	$1+i$	$2i$	$1+2i$	-1	0	$-1+i$	i	$-1+i$	i	$-1+2i$	$2i$
0	1	i	$1+i$	$1+i$	$2+i$	$1+2i$	$2+2i$	$-1+i$	i	$-1+2i$	$2i$	$2i$	$1+2i$	$3i$	$1+3i$
$l_1=0,$		$l_2=i$		$l_1=1,$		$l_2=i$		$l_1=i,$		$l_2=i$		$l_1=1+i,$		$l_2=i$	
0	i	-1	$-1+i$	0	i	-1	$-1+i$	0	i	-1	$-1+i$	0	i	-1	$-1+i$
0	i	-1	$-1+i$	1	$1+i$	0	i	i	$2i$	$-1+i$	$-1+2i$	$1+i$	$1+2i$	i	$2i$
0	i	-1	$-1+i$	i	$2i$	$-1+i$	$-1+2i$	-1	$-1+i$	-2	$-2+i$	$-1+i$	$-1+2i$	$-2+i$	$-2+2i$
0	i	-1	$-1+i$	$1+i$	$1+2i$	i	$2i$	$-1+i$	$-1+2i$	$-2+i$	$-2+2i$	$2i$	$3i$	$-1+2i$	$-1+3i$
$l_1=0,$		$l_2=1+i$		$l_1=1,$		$l_2=1+i$		$l_1=i,$		$l_2=1+i$		$l_1=1+i,$		$l_2=1+i$	
0	$1+i$	$-1+i$	$2i$	0	$1+i$	$-1+i$	$2i$	0	$1+i$	$-1+i$	$2i$	0	$1+i$	$-1+i$	$2i$
0	$1+i$	$-1+i$	$2i$	1	$2+i$	i	$1+2i$	i	$1+2i$	$-1+2i$	$3i$	$1+i$	$2+2i$	$2i$	$1+3i$
0	$1+i$	$-1+i$	$2i$	i	$1+2i$	$-1+2i$	$3i$	-1	i	$-2+i$	$-1+2i$	$-1+i$	$2i$	$-2+2i$	$-1+3i$
0	$1+i$	$-1+i$	$2i$	$1+i$	$2+2i$	$2i$	$1+3i$	$-1+i$	$2i$	$-2+2i$	$-1+3i$	$2i$	$1+3i$	$-1+3i$	$4i$

A simple case which might be considered is when l is a real integer, as in this case the factors $(-)^{\Sigma nl}$ all reduce to unity, and the l's in the characteristic may obviously be written with the plus sign.

PERIODS.

We may still, for simplicity, continue to deal with the double Ξ-functions, as the general properties which are being studied are capable of immediate generalization to the case of p-tuple functions. From the definition of these functions we see at once that if x_1 and x_2 denote any two integers we must have

32. $$\Xi\begin{pmatrix} l_1 & l_2 \\ \lambda_1 & \lambda_2 \end{pmatrix}(v_1+x_1,\ v_2+x_2)=\Xi\begin{pmatrix} l_1 & l_2 \\ \lambda_1+x_1 & \lambda_2+x_2 \end{pmatrix}(v_1,\ v_2),$$

that is, the effect of altering the arguments v_1 and v_2 by the addition of the integers x_1 and x_2 only interchanges the functions. If, however, we write $2x_1$ and $2x_2$ instead of x_1 and x_2 we have

$$33. \qquad \Xi\begin{pmatrix} l_1 & l_2 \\ \lambda_1 & \lambda_2 \end{pmatrix}(v_1 + 2x_1, v_2 + 2x_2) = \Xi\begin{pmatrix} l_1 & l_2 \\ \lambda_1 + 2x_1 & \lambda_2 + 2x_2 \end{pmatrix}(v_1, v_2)$$

or from equation 7,

$$34. \qquad \Xi\begin{pmatrix} l_1 & l_2 \\ \lambda_1 & \lambda_2 \end{pmatrix}(v_1 + 2x_1, v_2 + 2x_2) = (-)^{l_1x_1 + l_2x_2}\,\Xi\begin{pmatrix} l_1 & l_2 \\ \lambda_1 & \lambda_2 \end{pmatrix}(v_1, v_2)$$

and the factor $(-)^{l_1x_1+l_2x_2}$ has been shown to be either

$$+1, \ -1, \ +e^{-k\tau}, \ -e^{-k\tau},$$

where k is an integer; so the function is either unaltered, altered in sign, or multiplied by one of the constants $\pm e^{-k\tau}$. The cases where there is no alteration or at most a change of sign, correspond to l_1 and l_2 real integers. If we replace x_1 and x_2 by $4x_1$ and $4x_2$ respectively, x_1 and x_2 still integers, we have

$$35. \qquad \Xi\begin{pmatrix} l_1 & l_2 \\ \lambda_1 & \lambda_2 \end{pmatrix}(v_1 + 4x_1, v_2 + 4x_2) = (-)^{2(l_1x_1 + l_2x_2)}\,\Xi\begin{pmatrix} l_1 & l_2 \\ \lambda_1 & \lambda_2 \end{pmatrix}(v_1, v_2).$$

The factor $(-)^{2(l_1x_1+l_2x_2)}$ is in this case either $+1$ or $e^{-2k\pi}$, where k is an integer. There is thus in the case of the Ξ-functions no period in the ordinary sense of the word, those functions only possess a period for which l_1 and l_2 are real. There are $64, = 2^6$ functions which have periods, and $192, = 2^8 - 2^6$ which are multiplied by the above factor; in general there are 2^{3p} functions which possess a true period and $2^{3p}(2^p - 1)$ functions which are multiplied by an exponential factor. The *quasi-periods* are obtained just as in the case of the theta-functions, and to show their existence I shall merely reproduce, with the necessary changes of notation, Prof. Cayley's proof in §10 of his memoir. Taking x_1 and x_2 integers, we consider the effect of the change of v_1, v_2 into

$$v_1 + \frac{1}{\pi i}(a_{11}x_1 + a_{12}x_2), \ v_2 + \frac{1}{\pi i}(a_{12}x_1 + a_{22}x_2).$$

Starting from the function

$$\Xi\begin{pmatrix} l_1 - x_1 & l_2 - x_2 \\ \lambda_1 & \lambda_2 \end{pmatrix}\left(v_1 + \frac{1}{\pi i}(a_{11}x_1 + a_{12}x_2), \ v_2 + \frac{1}{\pi i}(a_{12}x_1 + a_{22}x_2)\right)$$

we have for the argument of the exponential

$$\tfrac{1}{4}(a_{11}, a_{12}, a_{22}\between n_1 + l_1 - x_1, n_2 + l_2 - x_2)^2$$
$$+ \tfrac{1}{2}\pi i\left\{(n_1 + l_1 - x_1)\left(v_1 + \lambda_1 + \frac{1}{\pi i}(a_{11}x_1 + a_{12}x_2)\right)\right.$$
$$\left. + (n_2 + l_2 - x_2)\left(v_2 + \lambda_2 + \frac{1}{\pi i}(a_{12}x_1 + a_{22}x_2)\right)\right\},$$

this is

$$\tfrac{1}{4}(a_{11}, a_{12}, a_{22} \between n_1 + l_1, n_2 + l_2)^2 + \tfrac{1}{2}\pi i((n_1 + l_1)(v_1 + \lambda_1) + (n_2 + l_2)(v_2 + \lambda_2))$$

(the original exponent) + other terms which are as follows: viz. they are

$$\begin{array}{ll} -\tfrac{1}{2}(a_{11}, a_{12}, a_{22} \between n_1 + l_1, n_2 + l_2 \between x_1, x_2) & +\tfrac{1}{2}[(n_1 + l_1)(a_{11}x_1 + a_{12}x_2) + (n_2 + l_2)(a_{12}x_1 + a_{22}x_2)] \\ +\tfrac{1}{4}(a_{11}, a_{12}, a_{22} \between x_1, x_2)^2 & -\tfrac{1}{2}\pi i[x_1(v_1 + \lambda_1) + x_2(v_2 + \lambda_2)] \\ & -\tfrac{1}{2}[x_1(a_{11}x_1 + a_{12}x_2) + x_2(a_{12}x_1 + a_{22}x_2)]. \end{array}$$

The terms in the right-hand column reduce at once to

$$\begin{aligned} &+\tfrac{1}{2}(a_{11}, a_{12}, a_{22} \between n_1 + l_1, n_2 + l_2 \between x_1, x_2) \\ &-\tfrac{1}{2}\pi i[x_1(v_1 + \lambda_1) + x_2(v_2 + \lambda_2)] \\ &-\tfrac{1}{2}[a_{11}, a_{12}, a_{22} \between x_1, x_2)^2. \end{aligned}$$

The original exponent is then increased by the terms

$$-\tfrac{1}{4}(a_{11}, a_{12}, a_{22} \between x_1, x_2)^2 - \tfrac{1}{2}\pi i[x_1(v_1 + \lambda_1) + x_2(v_2 + \lambda_2)],$$

which are independent of n_1 and n_2, and they thus affect each term of the series with the same exponential factor. The result is

$$\begin{aligned} &\Xi\begin{pmatrix} l_1 - x_1, & l_2 - x_2 \\ \lambda_1 & \lambda_2 \end{pmatrix}\left(v_1 + \frac{1}{\pi i}(a_{11}x_1 + a_{12}x_2), v_2 + \frac{1}{\pi i}(a_{12}x_1 + a_{22}x_2)\right) \\ 36.\quad &= \text{exp.}\{-\tfrac{1}{4}(a_{11}, a_{12}, a_{22} \between x_1, x_2)^2 - \tfrac{1}{2}\pi i[x_1(v_1 + \lambda_1) + x_2(v_2 + \lambda_2)]\}\,\Xi\begin{pmatrix} l_1 & l_2 \\ \lambda_2 & \lambda_2 \end{pmatrix}(v_1, v_2); \end{aligned}$$

or what is the same thing, replacing l_1 and l_2 by $l_1 + x_1$ and $l_2 + x_2$ (this change, since x_1 and x_2 are integers, only alters the real part of l_1 and l_2 and so is admissible),

$$\begin{aligned} &\Xi\begin{pmatrix} l_1, & l_2 \\ \lambda_1, & \lambda_2 \end{pmatrix}\left(v_1 + \frac{1}{\pi i}(a_{11}x_1 + a_{12}x_2), v_2 + \frac{1}{\pi i}(a_{12}x_1 + a_{22}x_2)\right. \\ 37.\quad &= \text{exp.}\,\{-\tfrac{1}{4}(a_{11}, a_{12}, a_{22} \between x_1, x_2)^2 - \tfrac{1}{2}\pi i[x_1(v_1 + \lambda_1) + x_2(v_2 + \lambda_2)]\} \\ &\qquad\qquad \Xi\begin{pmatrix} l_1 + x_1, & l_2 + x_2 \\ \lambda_1 & \lambda_2 \end{pmatrix}(v_1, v_2). \end{aligned}$$

If x_1 and x_2 are even, say $2x_1$ and $2x_2$, the function on the right-hand side of this equation becomes,

$$38.\quad \Xi\begin{pmatrix} l_1 + 2x_1, & l_2 + 2x_2 \\ \lambda_1, & \lambda_2 \end{pmatrix}(v_1, v_2), = \Xi\begin{pmatrix} l_2, & l_2 \\ \lambda_1, & \lambda_2 \end{pmatrix}(v_1, v_2),$$

but the exponential factor still remains. The formulæ show that the effect of changing v_1 and v_2 into

$$v_1 + \frac{1}{\pi i}(a_{11}x_1 + a_{12}x_2), \quad v_2 + \frac{1}{\pi i}(a_{12}x_1 + a_{22}x_2),$$

where x_1 and x_2 are integers, is to interchange the functions and to affect each of them with an exponential factor; and we may say, as in the case of the ϑ-functions, that $\frac{1}{\pi i}(a_{11}, a_{12})$, $\frac{1}{\pi i}(a_{12}, a_{22})$ are *conjoint quarter quasi-periods.* We may also say that the Ξ-functions have the quarter-periods $(1, 1)$, the half-periods $(2, 2)$ and the whole periods $(4, 4)$, understanding that only those functions are truly periodic for which the l's are all real, *i. e.* for one case all zero or unity.

THE PRODUCT-THEOREM.

If we multiply together two *theta-functions*, say

$$\vartheta\begin{pmatrix} l_1, & l_2 \\ \lambda_1, & \lambda_2 \end{pmatrix}(v_1 + v'_1,\ v_2 + v'_2),\quad \vartheta\begin{pmatrix} l'_1, & l'_2 \\ \lambda'_1, & \lambda'_2 \end{pmatrix}(v_1 - v'_1,\ v_2 - v'_2)$$

we have, for the result, the sum of four products of the form,

$$\Theta\begin{pmatrix} \frac{1}{2}(l_1+l'_1)+p_1, & \frac{1}{2}(l_2+l'_2)+p_2 \\ \lambda_1+\lambda'_1, & \lambda_2+\lambda'_2 \end{pmatrix}(2v_1,\ 2v_2).\Theta\begin{pmatrix} \frac{1}{2}(l_1-l'_1)+p_1, & \frac{1}{2}(l_2-l'_2)+p_2 \\ \lambda_1-\lambda'_1, & \lambda_2-\lambda'_2 \end{pmatrix}(2v'_1,\ 2v'_2),$$

where p_1 and p_2 take the values 0 and 1, and where Θ is written to denote that a_{11}, a_{12}, a_{22} are replaced by $2a_{11}$, $2a_{12}$, $2a_{22}$. The proof given of this theorem is wholly independent of the nature of l_1 and l_2 and so holds for the case of the double Ξ-functions. Using here ξ to denote that the parameters a_{11}, a_{12}, a_{22} have been changed into $2a_{11}$, $2a_{12}$, $2a_{22}$ we have

$$\text{39.}\qquad \Xi\begin{pmatrix} l_1, & l_2 \\ \lambda_1, & \lambda_2 \end{pmatrix}(v_1 + v'_1).\Xi\begin{pmatrix} l'_1, & l'_2 \\ \lambda'_1, & \lambda'_2 \end{pmatrix}(v_1 - v'_1)$$

$$= \xi\begin{pmatrix} \frac{1}{2}(l_1+l'_1), & \frac{1}{2}(l_2+l'_2) \\ \lambda_1+\lambda'_1, & \lambda_2+\lambda'_2 \end{pmatrix}(2v_1).\xi\begin{pmatrix} \frac{1}{2}(l_1-l'_1), & \frac{1}{2}(l_2-l'_2) \\ \lambda_1-\lambda'_1, & \lambda_2-\lambda'_2 \end{pmatrix}(2v'_1)$$

$$+ \xi\begin{pmatrix} \frac{1}{2}(l_1+l'_1)+1, & \frac{1}{2}(l_1+l'_2) \\ \lambda_1+\lambda'_1 & \lambda_2+\lambda'_2 \end{pmatrix}(2v_1).\xi\begin{pmatrix} \frac{1}{2}(l_1-l'_1)+1, & \frac{1}{2}(l_2-l'_2) \\ \lambda_1-\lambda'_1 & \lambda_2-\lambda'_2 \end{pmatrix}(2v'_1)$$

$$+ \xi\begin{pmatrix} \frac{1}{2}(l_1+l'_1), & \frac{1}{2}(l_2+l'_2)+1 \\ \lambda_1+\lambda'_1 & \lambda_2+\lambda'_2 \end{pmatrix}(2v_1).\xi\begin{pmatrix} \frac{1}{2}(l_1-l'_1), & \frac{1}{2}(l_2-l'_2)+1 \\ \lambda_1-\lambda'_1 & \lambda_2-\lambda'_2 \end{pmatrix}(2v'_1)$$

$$+ \xi\begin{pmatrix} \frac{1}{2}(l_1+l'_1)+1, & \frac{1}{2}(l_2+l'_2)+1 \\ \lambda_1+\lambda'_1, & \lambda_2+\lambda'_2 \end{pmatrix}(2v_1).\xi\begin{pmatrix} \frac{1}{2}(l_1-l'_1)+1, & \frac{1}{2}(l_2-l'_2)+1 \\ \lambda_1-\lambda'_1, & \lambda_2-\lambda'_2 \end{pmatrix}(2v'_1)$$

the arguments $v_1 + v'_1$, $v_1 - v'_1$ are written for brevity instead of $(v_1 + v'_1,\ v_2 + v'_2)$ and $(v_1 - v'_1,\ v_2 - v'_2)$, and similarly on the right-hand side of the equation one argument only is exhibited.

In order to develop the results it would be necessary, or at least desirable, to form two tables, one giving all the values of the upper line of the new characteristics, and the other giving all the values of the lower line of the same. This is what Prof. Cayley has done in his memoir on the double theta-functions. The tables are so long, however, that I do not give them here, particularly as their application to form a complete product table for the double Ξ-functions would give an exceedingly long table. There are in all 2^{16} (in general 2^{8p}) different products in the case of the double Ξ-functions; these consist of a *square-set* of 256 ($= 2^{4p}$) products and 255 other sets, each containing 256 products. Using suffixes for a moment to denote the different Ξ-functions, we define a *square-set* as a set consisting of products of the form,

$$\Xi_{ik}(v_1+v'_1,\ v_2+v'_2)\,.\,\Xi_{ik}(v_1-v'_1,\ v_2-v'_2)\,;$$

the remaining sets would have different suffixes for each Ξ.

All of the theorems which have been here set down for the double Ξ-functions are, as has been already remarked, capable of immediate generalization to the case of the p-tuple functions. An alteration of the l's by an even integer produces no change in the function; an alteration of the λ's by an even integer gives a formula precisely like that in the case of the ϑ-functions, the only difference being that the factor, say $(-)^{\delta}$, which is introduced, does not take merely the values $+1$ and -1, but the values $+1$, -1, $+e^{-\tau\pi}$, $-e^{-\tau\pi}$ where τ is an integer. An alteration of l or λ by an odd integer interchanges the functions (it would be interesting to examine the effect of increasing l or λ or both by a complex quantity; I shall take this up later). The change of v into $-v$ gives for 2^{2p} functions formulæ which are identical with those for the theta-functions. For the remaining $2^{2p}(2^{2p}-1)$ functions there is a factor under the sign of summation which depends for its value upon the value of n, and the factor outside the sign of summation is $+1$, -1, $-e^{-\tau\pi}$, $+e^{-\tau\pi}$.

By increasing the v's by $x_1, x_2 \ldots x_p$ respectively, we only interchange the functions (this of course is the same as increasing the λ's by the same integers); increasing each v by $2x$ we multiply the function by $(-)^{\Sigma lx}$, a factor which takes the values $+1$, -1, $-e^{-\tau\pi}$, $+e^{-\tau\pi}$; finally, increasing each v by $4x$ we multiply the function by $(-)^{2\Sigma lx}$, so that we have 2^{3p} functions which are unchanged by this alteration of the arguments v and $2^{3p}(2^p-1)$ which are multiplied by a factor $e^{-\tau\pi}$.

I shall proceed now to a more particular examination of the Ξ-functions for the case $p=1$. There are sixteen functions corresponding to this value of p, the characteristics for which are

$$\begin{matrix} \begin{pmatrix}0\\0\end{pmatrix} & \begin{pmatrix}1\\0\end{pmatrix} & \begin{pmatrix}0\\1\end{pmatrix} & \begin{pmatrix}1\\1\end{pmatrix} \\ \begin{pmatrix}i\\0\end{pmatrix} & \begin{pmatrix}1+i\\0\end{pmatrix} & \begin{pmatrix}i\\1\end{pmatrix} & \begin{pmatrix}1+i\\1\end{pmatrix} \\ \begin{pmatrix}0\\i\end{pmatrix} & \begin{pmatrix}1\\i\end{pmatrix} & \begin{pmatrix}0\\1+i\end{pmatrix} & \begin{pmatrix}1\\1+i\end{pmatrix} \\ \begin{pmatrix}i\\i\end{pmatrix} & \begin{pmatrix}1+i\\i\end{pmatrix} & \begin{pmatrix}i\\1+i\end{pmatrix} & \begin{pmatrix}1+i\\1+i\end{pmatrix} \end{matrix} \tag{40.}$$

The first line of 40 gives the ordinary theta-functions of one variable corresponding to the elliptic functions. For the first line we have $b=\beta=0$; for the second line $b=1$, $\beta=0$; for the third line $b=0$, $\beta=1$; for the fourth line $b=\beta=1$; in each of the four lines a and α take the values 0 and 1.

The Ξ-function of one variable is defined by the equation

$$\Xi\begin{pmatrix} l \\ \lambda \end{pmatrix}(v)=\Sigma e^{\frac{1}{4}\mathrm{a}(n+l)^2+\frac{1}{2}\pi i(n+l)(v+\lambda)}$$

where $l=a+ib$ and $\lambda=\alpha+i\beta$, a, b, α, β taking as above the values 0 and 1; a is also written instead of a_{11} in order to avoid suffixes. The 16 different functions will be denoted by the symbols

$$\begin{array}{cccc} \vartheta_1 & \vartheta_2 & \vartheta_3 & \vartheta_4 \\ \xi_1 & \xi_2 & \xi_3 & \xi_4 \\ \eta_1 & \eta_2 & \eta_3 & \eta_4 \\ \zeta_1 & \zeta_2 & \zeta_3 & \zeta_4 \end{array}$$

viz., we have, omitting for convenience the argument on the left-hand side,

$$\vartheta_1=\Xi\begin{pmatrix} 0 \\ 0 \end{pmatrix}(v),\quad \vartheta_2=\Xi\begin{pmatrix} 1 \\ 0 \end{pmatrix}(v),\quad \vartheta_3=\Xi\begin{pmatrix} 0 \\ 1 \end{pmatrix}(v),\quad \vartheta_4=\Xi\begin{pmatrix} 1 \\ 1 \end{pmatrix}(v),$$

$$\xi_1=\Xi\begin{pmatrix} i \\ 0 \end{pmatrix}(v),\quad \xi_2=\Xi\begin{pmatrix} 1+i \\ 0 \end{pmatrix}(v),\quad \xi_3=\Xi\begin{pmatrix} i \\ 1 \end{pmatrix}(v),\quad \xi_4=\Xi\begin{pmatrix} 1+i \\ 1 \end{pmatrix}(v),$$

$$\eta_1=\Xi\begin{pmatrix} 0 \\ i \end{pmatrix}(v),\quad \eta_2=\Xi\begin{pmatrix} 1 \\ i \end{pmatrix}(v),\quad \eta_3=\Xi\begin{pmatrix} 0 \\ 1+i \end{pmatrix}(v),\quad \eta_4=\Xi\begin{pmatrix} 1 \\ 1+i \end{pmatrix}(v),$$

$$\zeta_1=\Xi\begin{pmatrix} i \\ i \end{pmatrix}(v),\quad \zeta_2=\Xi\begin{pmatrix} 1+i \\ i \end{pmatrix}(v),\quad \zeta_3=\Xi\begin{pmatrix} i \\ 1+i \end{pmatrix}(v),\quad \zeta_4=\Xi\begin{pmatrix} 1+i \\ 1+i \end{pmatrix}(v).$$

Writing $e^{\mathrm{a}}=q$ and converting the exponentials into circular functions we have the known formulæ

$$\vartheta_1=1+2\sum_{r=1}^{r=\infty} q^{r^2}\cos r\pi v,\qquad \vartheta_2=2\sum_{r=0}^{r=\infty} q^{\frac{(2r+1)^2}{4}}\cos\frac{2r+1}{2}\pi v,$$

$$\vartheta_3=1+2\sum_{r=1}^{r=\infty}(-)^r q^{r^2}\cos r\pi v,\quad \vartheta_4=2\sum_{r=0}^{r=\infty}(-)^{2r+1}q^{\frac{(2r+1)^2}{4}}\sin\frac{2r+1}{2}\pi v.$$

Making the same substitution in the case of the functions ξ_1, ξ_2 . . . ζ_4, and writing $n=2r$ we have

$$\xi_1=e^{-\frac{\pi v}{2}}\sum_{r=0}^{r=\infty} q^{\frac{1}{4}(4r^2-1)}\{\cos\pi rv(q^{ri}+q^{-ri})+i\sin\pi rv(q^{ri}-q^{-ri})\}$$

$$\xi_2 = e^{-\frac{\pi v}{2}} \sum_{r=0}^{r=\infty} q^{r(r+1)} \left\{ \cos \frac{2r+1}{2} \pi v \left(q^{i\frac{2r+1}{2}} + q^{-i\frac{2r+1}{2}} \right) \right.$$
$$\left. + i \sin \frac{2r+1}{2} \pi v \left(q^{i\frac{2r+1}{2}} - q^{-i\frac{2r+1}{2}} \right) \right\}$$

$$\xi_3 = e^{-\frac{\pi(v+1)}{2}} \sum_{r=0}^{r=\infty} (-)^r q^{\frac{1}{4}(4r^2-1)} \{ \cos \pi r v (q^{ri} + q^{-ri}) + i \sin \pi r v (q^{ri} - q^{-ri}) \}$$

$$\xi_4 = i e^{-\frac{\pi(v+1)}{2}} \sum_{r=0}^{r=\infty} (-)^r q^{r(r+1)} \left\{ \cos \frac{2r+1}{2} \pi v \left(q^{i\frac{2r+1}{2}} + q^{-i\frac{2r+1}{2}} \right) \right.$$
$$\left. + i \sin \frac{2r+1}{2} \pi v \left(q^{i\frac{2r+1}{2}} + q^{i\frac{2r+1}{2}} \right) \right\}$$

$$\eta_1 = \sum_{r=0}^{r=\infty} q^{r^2} \{ \cos r\pi v (e^{r\pi} + e^{-r\pi}) - i \sin r\pi v (e^{r\pi} - e^{-r\pi}) \}$$

$$\eta_2 = \sum_{r=0}^{r=\infty} q^{\frac{1}{4}(2r+1)^2} \left\{ \cos \frac{2r+1}{2} \pi v \left(e^{\frac{2r+1}{2}\pi} + e^{-\frac{2r+1}{2}\pi} \right) \right.$$
$$\left. - i \sin \frac{2r+1}{2} \pi v \left(e^{\frac{2r+1}{2}\pi} - e^{-\frac{2r+1}{2}\pi} \right) \right\}$$

$$\eta_3 = \sum_{r=0}^{r=\infty} (-)^r q^{r^2} \{ \cos r\pi v (e^{r\pi} + e^{-r\pi}) - i \sin r\pi v (e^{r\pi} - e^{-r\pi}) \}$$

$$\eta_4 = i \sum_{r=0}^{r=\infty} (-)^r q^{\frac{1}{4}(2r+1)^2} \left\{ - \cos \frac{2r+1}{2} \pi v \left(e^{\frac{2r+1}{2}\pi} - e^{-\frac{2r+1}{2}\pi} \right) \right.$$
$$\left. + i \sin \frac{2r+1}{2} \pi v \left(e^{\frac{2r+1}{2}\pi} + e^{-\frac{2r+1}{2}\pi} \right) \right\}$$

$$\zeta_1 = - i e^{-\frac{\pi v}{2}} \sum_{r=0}^{r=\infty} e^{r\pi} q^{ri} q^{\frac{1}{4}(4r^2-1)} \{ \cos r\pi v (e^{-2r\pi} + q^{-2ri}) + i \sin r\pi v (e^{-2r\pi} - q^{-2ri}) \}$$

$$\zeta_2 = - i e^{-\frac{\pi v}{2}} \sum_{r=0}^{r=\infty} e^{\frac{(2r+1)\pi}{2}} . q^{\frac{(2r+1)i}{2}} . q^{r(r+1)} \left\{ \cos \frac{(2r+1)\pi v}{2} (e^{-(2r+1)\pi} + q^{-(2r+1)i}) \right.$$
$$\left. + i \sin \frac{(2r+1)\pi v}{2} (e^{-(2r+1)\pi} - q^{-(2r+1)i}) \right\}$$

$$\zeta_3 = - i e^{-\frac{\pi}{2}(v+1)} \sum_{r=0}^{r=\infty} (-)^r q^{\frac{1}{4}(4r^2-1)} e^{r\pi} q^{ri} \{ \cos r\pi v (e^{-2r\pi} + q^{-2ri})$$
$$+ i \sin r\pi v (e^{-2r\pi} - q^{-2ri}) \}$$

$$\zeta_4 = e^{-\frac{\pi}{2}(v+1)} \sum_{r=0}^{r=\infty} (-)^r q^{r(r+1)} e^{\frac{\pi}{2}(2r+1)} q^{\frac{1}{2}(2r+1)i} \left\{ \cos \frac{2r+1}{2} \pi v (e^{-\pi(2r+1)} - e^{-(2r+1)i}) \right.$$
$$\left. + i \sin \frac{2r+1}{2} \pi v (e^{-\pi(2r+1)} + e^{-(2r+1)i}) \right\}$$

The formulæ for ξ_1, ξ_2 . . . ζ_4 may be written in a little different form; viz., writing for q its value e^{a} we have

$$\xi_1 = 2e^{-\frac{\pi v}{2}} \sum_{r=0}^{r=\infty} q^{\frac{1}{4}(4r^2-1)} \cos r(\pi v + \mathrm{a})$$

$$\xi_2 = 2e^{-\frac{\pi v}{2}} \sum_{r=0}^{r=\infty} q^{r(r+1)} \cos \frac{2r+1}{2}(\pi v + \mathrm{a})$$

$$\xi_3 = 2e^{-\frac{\pi}{2}(v+1)} \sum_{r=0}^{r=\infty} (-)^r q^{\frac{1}{4}(4r^2-1)} \cos r(\pi v + \mathrm{a})$$

$$\xi_4 = -2e^{-\frac{\pi}{2}(v+1)} \sum_{r=0}^{r=\infty} (-)^r q^{r(r+1)} \sin \frac{2r+1}{2}(\pi v + \mathrm{a})$$

$$\eta_1 = 2 \sum_{r=0}^{r=\infty} q^{r^2} \cos r\pi(v - i)$$

$$\eta_2 = 2 \sum_{r=0}^{r=\infty} q^{\frac{1}{4}(2r+1)^2} . \cos \frac{2r+1}{2} \pi(v - i)$$

$$\eta_3 = 2 \sum_{r=0}^{r=\infty} (-)^r q^{r^2} \cos r\pi(v - i)$$

$$\eta_4 = -2 \sum_{r=0}^{r=\infty} (-)^r q^{\frac{1}{4}(2r+1)^2} \sin \frac{2r+1}{2} \pi(v - i)$$

$$\zeta_1 = -2ie^{-\frac{\pi v}{2}} \sum_{r=0}^{r=\infty} q^{\frac{1}{4}(4r^2-1)} \cos r(\pi(v+i) + \mathrm{a})$$

$$\zeta_2 = -2ie^{-\frac{\pi v}{2}} \sum_{r=0}^{r=\infty} q^{r(r+1)} \cos \frac{2r+1}{2}(\pi(v+i) + \mathrm{a})$$

$$\zeta_3 = -2ie^{\frac{\pi}{2}(v+1)} \sum_{r=0}^{r=\infty} (-)^r q^{\frac{1}{4}(4r^2-1)} \cos r(\pi(v+i) + \mathrm{a})$$

$$\zeta_4 = 2ie^{\frac{\pi}{2}(v+1)} \sum_{r=0}^{r=\infty} (-)^r q^{r(r+1)} \sin \frac{2r+1}{2}(\pi(v+i) + \mathrm{a})$$

The notation of hyperbolic functions might also have been employed for the representation of the functions $\eta_1, \eta_2 \ldots \zeta_4$. The constants derived from these functions by placing $v = 0$ are as follows:

$$\vartheta_1(0) = 1 + 2\sum_{r=0}^{r=\infty} q^{r^2}, \qquad \vartheta_2(0) = 2\sum_{r=0}^{r=\infty} q^{\frac{(2r+1)^2}{4}}$$

$$\vartheta_3(0) = 1 + 2\sum_{r=0}^{r=\infty} (-)^r q^{r^2}, \qquad \vartheta_4(0) = 0$$

$$\xi_1(0) = 2\sum_{r=0}^{r=\infty} q^{\frac{1}{4}(4r^2-1)} \cos r\mathrm{a}, \qquad \xi_2(0) = 2\sum_{r=0}^{r=\infty} q^{r(r+1)} \cos\frac{2r+1}{2}\mathrm{a}$$

$$\xi_3(0) = 2e^{-\frac{\pi}{2}}\sum_{r=0}^{r=\infty} (-)^r q^{\frac{1}{4}(4r^2-1)} \cos r\mathrm{a}, \qquad \xi_4(0) = -2e^{-\frac{\pi}{2}}\sum_{r=0}^{r=\infty} (-)^r q^{r(r+1)} \sin\frac{2r+1}{2}\mathrm{a}$$

$$\eta_1(0) = 2\sum_{r=0}^{r=\infty} q^{r^2} \cosh r\pi \qquad \eta_2(0) = 2\sum_{r=0}^{r=\infty} q^{\frac{1}{4}(2r+1)^2} \cosh\frac{2r+1}{2}\pi$$

$$\eta_3(0) = 2\sum_{r=0}^{r=\infty} (-)^r q^{r^2} \cosh r\pi \qquad \eta_4(0) = 2i\sum_{r=0}^{r=\infty} (-)^r q^{\frac{1}{4}(2r+1)^2} \sinh\frac{2r+1}{2}\pi$$

$$\zeta_1(0) = -2i\sum_{r=0}^{r=\infty} q^{\frac{1}{4}(4r^2-1)} \cos r(\pi i + \mathrm{a}), \quad \zeta_2(0) = -2i\sum_{r=0}^{r=\infty} q^{r(r+1)} \cos\frac{2r+1}{2}(\pi i + \mathrm{a})$$

$$\zeta_3(0) = -2ie^{\frac{\pi}{2}}\sum_{r=0}^{r=\infty} (-)^r q^{\frac{1}{4}(4r^2-1)} \cos r(\pi i + \mathrm{a}),$$

$$\zeta_4(0) = 2ie^{\frac{\pi}{2}}\sum_{r=0}^{r=\infty} (-)^r q^{r(r+1)} \sin\frac{2r+1}{2}(\pi i + \mathrm{a}).$$

The values of the functions ϑ_1, ϑ_2, ζ_3 and ζ_4 for $v = 0$ give rise (Prof. Cayley's memoir, page 912) to the constants,

$$k = \frac{\vartheta_2^2 0}{\vartheta_1^2 0}, \quad k' = \frac{\vartheta_3^2 0}{\vartheta_1^2 0}, \quad K = -\frac{\vartheta_1 0}{\vartheta_2 0}\cdot\frac{\vartheta_4' 0}{\vartheta_3 0}$$

or

$$k = 4\sqrt{q}\left\{\frac{1 + q^2 + q^6 + \ldots}{1 + 2q + 2q^4 + \ldots}\right\}^2, = 4\sqrt{q}(1 - 4q + 14q^2 + \ldots),$$

$$k' = \left\{\frac{1 - 2q + 2q^4 - \ldots}{1 + 2q + 2q^4 + \ldots}\right\}^2, = 1 - 8q + 32q^2 - 96q^3 + \ldots,$$

$$K = \frac{\pi(1 + 2q + 2q^4 + \ldots)(1 - 3q^2 + 5q^6 - \ldots)}{2(1 - 2q + 2q^4 - \ldots)(1 + q^2 + q^6 + \ldots)}, = \tfrac{1}{2}\pi(1 + 4q + 4q^2 + \ldots),$$

and identically $$k^2 + k'^2 = 1.$$

These are the ordinary constants of elliptic functions.

Similar relations exist for the ξ, η and ζ groups. Taking first the ξ-groups and making $v = 0$ we have

$$\begin{aligned} q^{\frac{1}{4}}\xi_1(0) &= 1 + 2q \cos \mathrm{a} + 2q^4 \cos 2\mathrm{a} + 2q^9 \cos 3\mathrm{a} + \ldots, = A_1, \\ q^{\frac{1}{4}}\xi_2(0) &= 2q^{\frac{1}{4}} \cos \tfrac{1}{2}\mathrm{a} + 2q^{\frac{9}{4}} \cos \tfrac{3}{2}\mathrm{a} + 2q^{\frac{25}{4}} \cos \tfrac{5}{2}\mathrm{a} + \ldots, = A_2, \\ e^{\frac{\pi}{2}} q^{\frac{1}{4}}\xi_3(0) &= 1 - 2q \cos \mathrm{a} + 2q^4 \cos 2\mathrm{a} - 2q^9 \cos 3\mathrm{a} + \ldots, = A_3, \\ e^{\frac{\pi}{2}} q^{\frac{1}{4}}\xi_4(0) &= -2q^{\frac{1}{4}} \sin \tfrac{1}{2}\mathrm{a} + 2q^{\frac{9}{4}} \sin \tfrac{3}{2}\mathrm{a} - 2q^{\frac{25}{4}} \sin \tfrac{5}{2}\mathrm{a} + \ldots, = A_4. \end{aligned}$$

If here a is real (and therefore negative), and if we write $\mathrm{a} = -2\pi$, the series obviously take the same form as those for $\vartheta(0)$; but without making any hypothesis as to the value of a if we write

$$h = \frac{A_2^2}{\sqrt{A_1^4 + A_4^4}}, \quad h' = \frac{A_3^2}{\sqrt{A_1^4 + A_4^4}},$$

we find readily the relation $\quad h^2 + h'^2 = 1,$

that is
$$A_2^4 + A_3^4 = A_1^4 + A_4^4,$$

or
$$q\xi_2^4(0) + e^{2\pi} q\xi_3^4(0) = q\xi_1^4(0) + e^{2\pi} q\xi_4^4(0),$$

or
$$\xi_2^4(0) + e^{2\pi}\xi_3^4(0) = \xi_1^4(0) + e^{2\pi}\xi_4^4(0).$$

Taking now the η-groups and making $v = 0$ and we have

$$\begin{aligned} \eta_1(0) &= 1 + 2q \cosh \pi + 2q^4 \cosh 2\pi + 2q^9 \cosh 3\pi + \ldots, \\ \eta_2(0) &= 2q^{\frac{1}{4}} \cosh \tfrac{1}{2}\pi + 2q^{\frac{9}{4}} \cosh \tfrac{3}{2}\pi + 2q^{\frac{25}{4}} \cosh \tfrac{3}{2}\pi, \\ \eta_3(0) &= 1 - 2q \cosh \pi + 2q^4 \cosh 2\pi - 2q^9 \cosh 3\pi, \\ \eta_4(0) &= i\{-2q^{\frac{1}{4}} \sinh \tfrac{1}{2}\pi + 2q^{\frac{9}{4}} \sinh \tfrac{3}{2}\pi - 2q^{\frac{25}{4}} \sinh \tfrac{5}{2}\pi + \ldots\} \end{aligned}$$

Writing
$$g = \frac{\eta_2^2(0)}{\sqrt{\eta_1^4(0) + \eta_4^4(0)}}, \quad g' = \frac{\eta_3^2(0)}{\sqrt{\eta_1^4(0) + \eta_4^4(0)}},$$

we have
$$g^2 + g'^2 = 1,$$

or
$$\eta_2^4(0) + \eta_3^4(0) = \eta_1^4(0) + \eta_4^4(0).$$

Finally, taking the ζ-groups and making $v = 0$ we have

$$\begin{aligned} q^{\frac{1}{4}}\zeta_1(0) &= 1 + 2q \cos(\mathrm{a} + \pi i) + 2q^4 \cos 2(\mathrm{a} + \pi i) + \ldots, = B_1, \\ q^{\frac{1}{4}}\zeta_2(0) &= 2q^{\frac{1}{4}} \cos \tfrac{1}{2}(\mathrm{a} + \pi i) + 2q^{\frac{9}{4}} \cos \tfrac{3}{2}(\mathrm{a} + \pi i) + \ldots, = B_2, \\ e^{\frac{\pi}{2}} q^{\frac{1}{4}}\zeta_3(0) &= 1 - 2q \cos(\mathrm{a} + \pi i) + 2q^4 \cos 2(\mathrm{a} + \pi i) - \ldots, = B_3, \\ e^{\frac{\pi}{2}} q^{\frac{1}{4}}\zeta_4(0) &= -2q^{\frac{1}{4}} \sin \tfrac{1}{2}(\mathrm{a} + \pi i) + 2q^{\frac{9}{4}} \sin \tfrac{3}{2}(\mathrm{a} + \pi i) - \ldots, = B_4. \end{aligned}$$

Writing
$$F = \frac{B_2^2}{\sqrt{B_1^4 + B_4^4}}, \quad F' = \frac{B_3^2}{\sqrt{B_1^4 + B_4^4}},$$

we obtain
$$F^2 + F'^2 = 1,$$

or
$$\zeta_2^4(0) + e^{2\pi}\zeta_3^4(0) = \zeta_1^4(0) + e^{2\pi}\zeta_4^4(0).$$